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Credibilistic Risk Aversion

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In the probabilistic risk aversion approach, risks are presumed as random variables with known probability distributions. However, in some practical cases, for example, due to the absence of historical data, the inherent uncertain characteristic of risks or different subject judgements from the decision makers may be hard or not appropriate to be estimated with probability distributions. Therefore the traditional probabilistic risk aversion theory is ineffective. Thus, in order to deal with these cases, we suggest to measure those kinds of risks as fuzzy variables, and accordingly to present an alternative risk aversion approach by employing the credibility theory. In the present paper, first, the definition of credibilistic risk premium proposed by Georgescu and Kinnunen (2013) is revised by taking the initial wealth into consideration, and provide a general method to compute the credibilistic risk premium. Secondly, regarding the risks represented with the commonly used LR fuzzy intervals, a simple calculation formula for the local credibilistic risk premium is put forward. Finally, in a global sense, several equivalent propositions for the comparative risk aversion under the credibility measurement are provided. Illustrated examples are presented to show the applicability of the theoretical findings.

Keywords: Risk aversion, LR fuzzy interval, Credibility theory, Credibilistic risk premium

JEL Classification: G1; D81; G12

1. Introduction

The *expected utility theory*, which has been proposed by von Neumann and Morgenstern (1944), is well acknowledged and accepted as a normative framework of decision making under risk when certain axioms of rational behaviour are applied for. Under this perspective, one of the basic mathematical assumptions is that the utility function, $u(x)$, is strictly increasing and concave. This implies that when an individual confronts a certain income (or loss), and an uncertain profit (or risk), if the value of the certain one is equal to the expected value of the uncertain one, he would prefer the former. This kind of phenomenon is known as *risk aversion*. Afterwards, on the foundation of the expected utility theory, the concept of *risk premium* was initially defined by Pratt (1964) and Arrow (1970), and since then, it is utilized to interpret (rational) behaviours in economics under different types of risks.

In the existing literature of finance, many theoretical studies have been confronted on the basis of risk aversion and expected utility theory. Just to mention some, Menezes and Hanson (1970) reveal the economic significance of the absolute, relative, and partial relative risk aversion functions when one of the wealth and risk parameters remains fixed, and the other varies. Loomes and Sugden (1982) consider that the expected utility theory provides a restrictive definition for rational

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behaviours, and therefore they propose the *regret* theory as an alternative approach. Later, Bell (1985) and Gul (1991) with his axiomatic approach, explore the influence of the disappointment aversion, which is a related concept with regret, and they integrate it into the utility theory by comparing the actual benefits and prior expectations. Besides, Rabin (2000) gives an interesting discussion on the implications of theorems of risk aversion and expected utility theory. Furthermore, the risk aversion theory has been extensively applied in behavioural economics; just to mention a few applications: consider the theory of the firm (Sandmo 1971), the asset return and the consumption (Hansen and Singleton 1983), the optimal portfolio choice (Dow and da Costa Werlang 1992), the lottery choice (Holt *et al.* 2002), the tax evasion (Hashimzade *et al.* 2013), and the adoption of new technologies (Barham *et al.* 2014).

So far, the developments on risk aversion have been based on a standard probabilistic framework by expressing risks as random variables with known probability distributions, which are assumed to derive from observing the uncertain events or the historical data. However, the above mentioned conditions may not be always occur. For instance, it is hard to give a probability distribution for the pricing evaluation of an *antique*, or for a *new stock*, since there are no historical data to be used in order to get the exact probability distribution of the price. Let alone that the behavioural attitude of the investor, i.e., educational background, emotional factors, culture, etc., can give different estimates on the risk, and it can make a final reflection on the individual's decision on the risk. Consequently, the traditional probability distribution might not be adaptable to deal with the uncertain information about the risk. Therefore, many researchers keep making efforts to conquer this aporia in risk aversion by other theories, like *prospect* theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992), *possibility* theory (Zadeh 1979, Georgescu 2009, 2012), and *uncertainty* theory (Liu 2002, Zhou *et al.* 2015). As it will be clearer in the next paragraphs, in the present paper, we focus our attention on the possibility and uncertainty theory. Thus, further discussion around the prospect theory is omitted.

Possibility theory is introduced by Zadeh (1979), as an alternative to probability theory, to deal with certain types of uncertainty. This approach has been accepted gradually and employed successfully to handle cases that the distribution of the uncertain events is unknown or the uncertainty is formed by human thoughts. By taking both the possibility and the necessity of a fuzzy set into consideration, the credibility theory was established by Liu (2002), and now it is a widely exploited approach to calculate the expected value and the variance of a fuzzy set owing to the property of self-dual of the credibility measure.

In this paper, we assume that the risk can not be expressed using a random variable, and thus, we suggest an alternative formulation under certain types of uncertainty. Thus, we utilize a fuzzy variable to represent the risk and proceed the subsequent studies on the basis of the credibility theory. To sum up the main innovations of our approach, first, we revise the concept of credibilistic risk premium presented initially by Georgescu and Kinnunen (2013), which overlook the influence of the initial assets on the attitude of the decision maker. Secondly, based on the operational law with respect to the commonly used LR[1] fuzzy intervals, see Zhou and Zhao (2016), we are able to calculate the local credibilistic risk premium. Furthermore, a comparative risk aversion in a global sense is also put forward.

The rest of this paper is organized as follows. Section 2 reviews the fundamental background on probabilistic risk aversion and necessary definitions on the credibility theory. Section 3 presents the main contributions of our work including the revised concept of credibilistic risk premium, a general calculation for the credibilistic risk premium, the simple calculation for the local credibilistic risk premium, and some results for the comparative risk aversion. Section 4 gives three illustrated examples to show the applicability of the theoretical results and the usefulness of our approach. Section 5 concludes the paper. Ideas for future research are also elaborated.

[1]i.e., They are L-R type fuzzy intervals with any left and right spreads. Initially, they have been introduced by Dubois and Prade (1981).

2. Preliminaries

On the ground of the expected utility theory proposed by the seminal work of von Neumann and Morgenstern (1944), and implemented by Pratt (1964) and Arrow (1970), the risk aversion presuming the risk is a random variable, see also Laffont (1989). In this section, initially we review briefly some well known concepts and results with respect to the probabilistic risk aversion. Then, we introduce the possibility as well as the credibility theories which provide the foundation of risk aversion under the credibility measure.

2.1. *Probabilistic risk premium*

Assume that the decision maker has the asset, x, and its utility function is given by $u(x)$. The concept of risk premium, π, initially proposed by Pratt (1964) is interpreted as the offset that makes the indifference between obtaining a risk $\tilde{z}$ assessed with the probability measurement and obtaining the exact amount $E(\tilde{z}) - \pi$, where $E(\tilde{z})$ is the expected value of $\tilde{z}$. Pratt (1964, 1992) gives the expression for this relationship of equivalence as follows,

$$u\big(x + E(\tilde{z}) - \pi(x, \tilde{z})\big) = E\big(u(x + \tilde{z})\big), \tag{1}$$

where $E(\tilde{z}) - \pi(x, \tilde{z})$ denotes the cash equivalent of uncertain risk $\tilde{z}$, and the form of $\pi(x, \tilde{z})$ shows that the risk premium depends on x and $\tilde{z}$. Since the utility function, $u(x)$, is always assumed strictly increasing, its inverse function, $u^{-1}(x)$, exists, and Eq. (1) can be converted into

$$\pi(x, \tilde{z}) = x + E(\tilde{z}) - u^{-1}\Big(E\big(u(x + \tilde{z})\big)\Big), \tag{2}$$

which is a general measure for the risk premium $\pi(x, \tilde{z})$. If for any x and $\tilde{z}$, $\pi(x, \tilde{z}) \geq 0$, then we call the utility function or the decision maker possessing it risk-aversion globally, which is actually equivalent with the concavity of $u(x)$, i.e., $u''(x) \leq 0$. Subsequently, at the status of assets x, $\pi(x, \tilde{z})$ for a small $\tilde{z}$, where "*small*" means that the variance of $\tilde{z}$, $\sigma_z^2 \to 0$, is considered as the *local* risk premium at the point x for the decision maker. Then, through Taylor formula, a simple calculation for the local risk premium depending on the utility function, and the variance of uncertain risk can be provided, see Pratt (1964, 1992), as follows

$$\begin{aligned} \pi(x, \tilde{z}) &= -\frac{1}{2}\sigma_z^2 \frac{u''(x + E(\tilde{z}))}{u'(x + E(\tilde{z}))} + o(\sigma_z^2) \\ &\approx \frac{1}{2}\sigma_z^2 r\big(x + E(\tilde{z})\big), \end{aligned} \tag{3}$$

where $r(x)$ is the commonly used local risk aversion function in the utility theory, and

$$r(x) = -\frac{u''(x)}{u'(x)}. \tag{4}$$

Indeed, the computational aspects for the calculation of the local risk premium are simple and straightforward, since they are based on the local risk aversion function, $r(x)$. However, a comparative risk aversion globally can also be obtained. Particularly, let us assume that $u_1(x)$ and $u_2(x)$ are utility functions of two decision makers. According to Eq. (4), the local risk aversion functions $r_1(x)$ and $r_2(x)$ can derive immediately. For any x, if the two local risk aversion functions satisfy $r_1(x) \geq r_2(x)$, Pratt (1964, 1992) proves the following theorem which illustrates that in that case, i.e., for every risk $\tilde{z}$ regardless of the value of its variance, the decision maker with $u_1(x)$ is globally more risk averse than the other.

Theorem 1 (Pratt 1964) Let $r_i(x)$ and $\pi_i(x, \tilde{z})$ be the local risk aversions and risk premiums corresponding to utility functions $u_i, i = 1, 2$, respectively. The following propositions are equivalent:
(i) $r_1(x) \geq r_2(x)$ for any x;
(ii) $\pi_1(x, \tilde{z}) \geq \pi_2(x, \tilde{z})$ for any x and $\tilde{z}$;
(iii) $u_1(u_2^{-1}(x))$ is concave.

Regarding the decision makers whose local risk aversion function $r(x)$ is decreasing, an extended theorem is deduced directly as follows.

Theorem 2 (Pratt 1964) The following propositions are equivalent:
(i') $r(x)$ decreases with x;
(ii') $\pi(x, \tilde{z})$ decreases with x for any $\tilde{z}$;
(iii') $u'(u^{-1}(x))$ is convex.

For the risk premium, $\pi(x, \tilde{z})$ that is decreasing with x for any $\tilde{z}$, which means that with the increasing of the assets x, the decision maker is willing more to take the uncertain risk $\tilde{z}$. Pratt (1964) names the corresponding utility function as a *decreasing risk aversion globally.*

2.2. *Possibility theory*

As it has been mentioned in the introduction, the possibility measure was initially proposed by Zadeh (1979), and it is used as a measurement for fuzzy sets. As a different approach of the probability theory, the possibility theory possesses its own distinctive adaptability and applicability in dealing with the uncertain problems that the distribution of the included uncertain event is tough to be given or the subjective thoughts make an influence on the estimation of that. In recent years, the possibility theory has greatly changed the way that ambiguity and imprecision were conventionally considered and got a notable development.

For the better understanding of what if follows, the following notation needs to be referred. Let a nonempty set Θ represent the sample space, and $\mathcal{P}(\Theta)$ the power set of Θ. Assign to each event A a number $\text{Pos}\{A\}$ indicating the possibility that A will occur. Then, for any collection of events in $\mathcal{P}(\Theta)$, we have that

$$\text{Pos}\{\cup_i A_i\} = \sup_i \text{Pos}\{A_i\}, \tag{5}$$

which means that the possibility degree of a set of uncertain events is equal to the maximal possibility degree of all the uncertain events in that set.

On the other hand, the necessity $\text{Nec}\{A\}$ is also a measurement of a fuzzy event A, see Zadeh (1979), which is defined as the impossibility of the opposite set A^c, i.e.,

$$\text{Nec}\{A\} = 1 - \text{Pos}\{A^c\}. \tag{6}$$

Example 1: As a simple illustration of Eqs. (5) and (6), assume that there is an investment A, and its outcome after one month is uncertain, whose possible values and corresponding possibility degrees are provided in Table 1. Given the assumption that this investment will make profits, i.e., the set of collection of $\{A_3, A_4, A_5\}$, and its possibility degree is equal to $\text{Pos}\{A_3, A_4, A_5\}$. Thus, according to Eq. (5), we have $\text{Pos}\{A_3, A_4, A_5\} = \sup_{i=3,4,5} \text{Pos}\{A_i\} = 1$. Similarly, from Eq. (6), the necessity degree of the uncertain event making profits is given by $\text{Nec}\{A_3, A_4, A_5\} = 1 - \text{Pos}\{A_1, A_2\} = 1 - \sup_{i=1,2} \text{Pos}\{A_i\} = 1 - 0.7 = 0.3$. Based on the possibility measure, the definition of the membership function is given, which is used to express a fuzzy variable directly.

Definition 1 Let ξ be a fuzzy variable which is defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$.

Table 1. Possibility distribution of the uncertain investment A

i	1	2	3	4	5
A_i	-100	-50	100	150	200
$\mathrm{Pos}\{A_i\}$	0.5	0.7	1	0.6	0.2

Then, its membership function is derived from the possibility measure by

$$\mu(x) = \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\}, \quad x \in \Re, \tag{7}$$

where $\Re$ is the set of real numbers.

Afterwards, the well-known LR fuzzy interval is presented which is widely employed in real world applications, see for more details, Dubois and Prade (1981, 2013).

Definition 2 (Dubois and Prade 2013) A shape function L (or similarly for R) is a function from $\Re^+ \to [0,1]$ such that
(1) $L(0) = 1$;
(2) $L(x) < 1, \ \forall x > 0$;
(3) $L(x) > 0, \ \forall x < 1$;
(4) $L(1) = 0$ [*or* $L(x) > 0, \ \forall x$ *and* $L(+\infty) = 0$];
(5) $L(x)$ is decreasing on the open interval $\{x|0 < L(x) < 1\}$.

Definition 3 (Dubois and Prade 2013) A fuzzy interval $\widetilde{M}$ is of LR-type if there exist shape functions L (for left), R (for right) and four parameters $(\underline{m}, \overline{m}) \in \Re^2 \bigcup \{-\infty, +\infty\}, \alpha > 0, \beta > 0$ with membership function

$$\mu(x) = \begin{cases} L\left(\dfrac{\underline{m} - x}{\alpha}\right), & \text{if } x < \underline{m} \\ \quad 1, & \text{if } \underline{m} \le x \le \overline{m} \\ R\left(\dfrac{x - \overline{m}}{\beta}\right), & \text{if } x > \overline{m}, \end{cases} \tag{8}$$

and the fuzzy interval is denoted by $\widetilde{M} = (\underline{m}, \overline{m}, \alpha, \beta)_{LR}$.

2.3. *Credibility theory*

Considering the overestimate and the underestimate of the possibility and the necessity for uncertain events, their average is suggested to be the most reasonable measure, see Liu and Liu (2002), and it is defined to be the *credibility measure*, Cr. Then $\mathrm{Cr}\{A\}$ indicates the credibility that an uncertain event A will occur, i.e.,

$$\mathrm{Cr}\{A\} = \frac{1}{2}(\mathrm{Pos}\{A\} + \mathrm{Nec}\{A\}). \tag{9}$$

Consequently, the credibility distribution of a fuzzy variable is given by the following definition, see Liu (2002).

Definition 4 (Liu 2002) The credibility distribution $\Phi : [-\infty, +\infty] \to [0,1]$ of a fuzzy variable ξ

is defined by

$$\Phi(x) = \mathrm{Cr}\{\theta \in \Theta \mid \xi(\theta) \le x\}. \tag{10}$$

Then, combining Eqs. (5)-10), the credibility distribution of a fuzzy variable is calculated by the use of its membership function as follows.

Theorem 3 (Liu and Gao 2007) Let ξ be one fuzzy variable with membership function μ. Its credibility distribution is calculated by

$$\Phi(x) = \frac{1}{2}\left(\sup_{y\le x}\mu(y) + 1 - \sup_{y>x}\mu(y)\right), \quad \forall x \in \Re. \tag{11}$$

From Eq. (11), and Theorem 3, it can be shown that the credibility distribution of one fuzzy variable is non-decreasing. Moreover, the definition of the independence of fuzzy variables is presented by Liu and Gao (2007), and it is related to the credibility measure as well as with an equivalent theorem. In more details, let us use the following definition.

Definition 5 (Liu and Gao 2007) The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \min_{1\le i\le n}\mathrm{Cr}\{\xi_i \in B_i\}, \tag{12}$$

for any sets $B_1, B_2, \cdots, B_n$ of $\Re$.

Then, the following necessary and sufficient condition is provided by a known theorem.

Theorem 4 (Liu and Gao 2007) The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if

$$\mathrm{Cr}\left\{\bigcup_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \max_{1\le i\le n}\mathrm{Cr}\left\{\xi_i \in B_i\right\}, \tag{13}$$

for any sets $B_1, B_2, \cdots, B_n$ of $\Re$.

Before, we proceed further, an illustrated example is provided for the better understanding of the notion of independence for the fuzzy variables.

Example 2: Assume that there are two uncertain investments ξ_1 and ξ_2, and both of them have three possible values. Their respective possibility distributions and their joint possibility distribution are listed in Table 2.

Table 2. Possibility distributions of ξ_1, ξ_2 and their joint possibility distribution

$\mathrm{Pos}\{(\xi_1, \xi_2)\}$	ξ_1^1	ξ_1^2	ξ_1^3	$\mathrm{Pos}\{\xi_2 = \xi_2^j\}$
ξ_2^1	0.3	0.4	0.7	0.8
ξ_2^2	0.6	1.0	0.9	1.0
ξ_2^3	0.4	0.5	0.3	0.6
$\mathrm{Pos}\{\xi_1 = \xi_1^i\}$	0.7	1.0	0.9	

Let $B_1 = \{\xi_1^1\}, B_2 = \{\xi_2^1\}$. Then we know $\mathrm{Cr}\{\xi_1^1 \cap \xi_2^1\} = \dfrac{1}{2}(\mathrm{Pos}\{\xi_1^1 \cap \xi_2^1\} + \mathrm{Nec}\{\xi_1^1 \cap \xi_2^1\}) = 0.15,$

while $\mathrm{Cr}\{\xi_1^1\} = 0.35$ and $\mathrm{Cr}\{\xi_2^1\} = 0.4$, which means that there exists at least one set B_1 such that $\mathrm{Cr}\{\xi_1^1 \cap \xi_2^1\} \neq \min\{\mathrm{Cr}\{\xi_1^1\}, \mathrm{Cr}\{\xi_2^1\}\}$. Thus ξ_1 and ξ_2 are not independent.

Additionally, in this subsection, the notion of the inverse credibility distribution is given, see Zhou and Zhao (2016).

Definition 6 (Zhou and Zhao 2016) Let $\xi = (\underline{m}, \overline{m}, \alpha, \beta)_{LR}$ be an LR fuzzy interval with the credibility distribution Φ. The inverse function Φ^{-1} with $\Phi^{-1}(0.5) = a$ ($\forall a \in [\underline{m}, \overline{m}]$), is called the inverse credibility distribution of ξ.

Note that the inverse credibility distribution Φ^{-1} is well defined in the interval $(0, 1)$. If it is required, it can extend its domain via the following expansion:

$$\Phi^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha) \quad \text{and} \quad \Phi^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha). \tag{14}$$

Furthermore, it should be mentioned here that Zhou and Zhao (2016) extend the study provided by Zhou *et al.* (2015) which gives an operational law for independent LR fuzzy intervals.

Theorem 5 (Zhou and Zhao 2016) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent LR fuzzy intervals with credibility distributions Φ_1, $\Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then

$$\xi = f(\xi_1, \cdots, \xi_m, \xi_{m+1}, \cdots, \xi_n) \tag{15}$$

is an LR fuzzy interval with inverse credibility distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{16}$$

Now, what is more, Liu and Liu (2002) put forward the general definition of the expected value operator for fuzzy variables on the basis of their credibility measure.

Definition 7 (Liu and Liu 2002) Let ξ be a fuzzy variable. Then, the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r, \tag{17}$$

provided that at least one of the two integrals is finite.

If ξ is an LR fuzzy interval with credibility distribution Φ, Eq. (17) is equivalent to

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha, \tag{18}$$

where $\Phi^{-1}(\alpha)$ is the inverse credibility distribution of ξ. Besides, let ξ and η be independent LR fuzzy intervals with finite expected values, then $\forall a, b \in \Re$,

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{19}$$

Moreover, for some special functions of fuzzy variables, e.g., convex functions, by just using the known Jensen's inequality, Liu (2002) proves a related inequality.

Theorem 6 (Liu 2002) Let ξ be a fuzzy variable, and f a convex function. If $E[\xi]$ and $E[f(\xi)]$ exist and are finite, then

$$f(E[\xi]) \leq E[f(\xi)].$$

Accordingly, a very useful corollary can derive straightforwardly, which is used in the proof of the next section.

Corollary 1 Let ξ be a fuzzy variable, and f a concave function. If $E[\xi]$ and $E[f(\xi)]$ exist and are finite, then

$$f(E[\xi]) \geq E[f(\xi)].$$

In addition, the variance of a fuzzy variable ξ is defined by Liu (2002).

Definition 8 (Liu and Liu 2002) Let ξ be a fuzzy variable with finite expected value $E[\xi]$. The variance of ξ is defined as

$$V[\xi] = E[(\xi - E[\xi])^2]. \tag{20}$$

Finally, for an LR fuzzy interval ξ with credibility distribution Φ, and finite expected value $E[\xi]$, Zhou and Zhao (2016) prove that

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - E[\xi])^2 \mathrm{d}\alpha, \tag{21}$$

and

$$V[a\xi + b] = a^2 V[\xi], \quad \forall a, b \in \Re. \tag{22}$$

3. Risk Aversion under Credibility Theory

In this section, we use the credibility theory for the analysis of risk aversion, and the credibilistic risk aversion is proposed as an alternative of the probabilistic risk aversion.

3.1. *Credibilistic risk premium*

In the literature of fuzzy sets and topics related to risk theory, Georgescu and Kinnunen (2013) gave a definition for the credibilistic risk premium without the consideration of the initial assets. In our approach, an amendment of their definition is provided. Assume that a decision maker with assets x and a utility function $u(x)$ faces an imprecise risk, the distribution of which is assumed to be the fuzzy variable ξ. Then, the credibilistic risk premium $\lambda(x, \xi)$ might be interpreted as an offset that makes the imprecise risk ξ and the certain amount $E[\xi] - \lambda$ to be indifferent. The credibilistic risk premium is a function of initial assets x and the credibility distribution of uncertain risk ξ. Definition 9 is useful in what it follows.

Definition 9 The *credibilistic risk premium* $\lambda(x, \xi)$ of ξ with respect to the utility function $u(x)$ is defined as

$$u(x + E[\xi] - \lambda(x, \xi)) = E[u(x + \xi)], \tag{23}$$

where x is the initial asset, ξ is a fuzzy variable representing the uncertain risk, and $E[\xi]$ and $E[u(x+\xi)]$ are the expected values of ξ and $u(x+\xi)$ which are calculated by utilizing the credibility measurement, respectively.

Based on Eq. (23), the credibilistic risk premium $\lambda(x+a, \xi-a)$ with respect to assets $x+a$ and an imprecise risk $\xi - a$, where a is a constant number, satisfies

$$u(x + a + E[\xi - a] - \lambda(x + a, \xi - a)) = E[u(x + a + \xi - a)]. \tag{24}$$

Since for any fuzzy variable, $E[\xi - a] = E[\xi] - a$, thus Eq. (24) is equal to

$$u(x + E[\xi] - \lambda(x + a, \xi - a)) = E[u(x + \xi)]. \tag{25}$$

Through Eqs. (23) and (25), we obtain

$$\lambda(x, \xi) = \lambda(x + a, \xi - a). \tag{26}$$

Besides, similar to the probabilistic risk premium, the credibilistic risk premium could also be derived directly through the operation of an inverse function of u that is

$$\lambda(x, \xi) = x + E[\xi] - u^{-1}\big(E[u(x + \xi)]\big). \tag{27}$$

Furthermore, letting $\lambda_a(x, \xi) = E[\xi] - \lambda(x, \xi)$, then Eq. (23) can be converted to

$$u(x + \lambda_a(x, \xi)) = E[u(x + \xi)], \tag{28}$$

where $\lambda_a(x, \xi)$ can be seen as the cash equivalent of ξ, i.e., the smallest amount that the decision maker would like to sell ξ. From the perspective of buyer, letting $\lambda_b(x, \xi)$ be the largest amount that he would like to buy ξ, then the indifference in his mind can be interpreted as

$$u(x) = E[u(x - \lambda_b(x, \xi) + \xi)]. \tag{29}$$

3.2. *Local risk aversion under credibility measure*

In theoretical studies as well as in applications of fuzzy sets theory, triangular and trapezoidal, and some other kinds of LR fuzzy intervals are the most commonly used fuzzy variables. In this part of the paper, we focus our attention on the family of risks which are denoted with the LR fuzzy intervals, and present a simple formula to calculate the local credibilistic risk premium.

We consider an actually neutral and small risk ξ, i.e., $E[\xi] = 0$, $V[\xi] \to 0$, then Eq. (23) can be converted to

$$u(x - \lambda(x, \xi)) = E[u(x + \xi)]. \tag{30}$$

By applying the Taylor formula of the second degree, expanding u around x on both sides of Eq. (30), we obtain

$$u(x - \lambda) = u(x) - \lambda u'(x) + O(\lambda^2) \tag{31}$$

and

$$E[u(x + \xi)] = E[u(x) + \xi u'(x) + \frac{1}{2}\xi^2 u''(x) + O(\xi^3)]. \tag{32}$$

It can be assumed that the utility function $u(x)$ is strictly increasing and ξ is an LR fuzzy interval, thus based on Theorem 5, for any $\alpha \in [0,1]$, the inverse credibility distribution of $u(x+\xi)$ in Eq. (32), which is defined by Ψ^{-1}, can be derived from the inverse credibility distribution of ξ, and it is denoted by Φ^{-1} as follows,

$$\begin{aligned}\Psi^{-1}(\alpha) &= u(x+\Phi^{-1}(\alpha)) \\ &= u(x)+\Phi^{-1}(\alpha)u'(x)+\frac{u''(x)}{2}(\Phi^{-1}(\alpha))^2+O\big((\Phi^{-1}(\alpha))^3\big).\end{aligned} \tag{33}$$

Thus, considering Eqs. (18), (21), (31), and (33), the expected value of $u(x+\xi)$ in Eq. (32) is calculated by the following expression.

$$\begin{aligned}E[u(x+\xi)] &= \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 u(x+\Phi^{-1}(\alpha))\mathrm{d}\alpha \\ &= \int_0^1 \left[u(x)+\Phi^{-1}(\alpha)u'(x)+\frac{u''(x)}{2}(\Phi^{-1}(\alpha))^2+O\big((\Phi^{-1}(\alpha))^3\big)\right]\mathrm{d}\alpha \\ &= u(x)+u'(x)\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha+\frac{u''(x)}{2}\int_0^1 (\Phi^{-1}(\alpha))^2\mathrm{d}\alpha+\int_0^1 O\big((\Phi^{-1}(\alpha))^3\big)\mathrm{d}\alpha \\ &= u(x)+u'(x)E[\xi]+\frac{u''(x)}{2}V[\xi]+o(V[\xi]) \\ &= u(x)+\frac{u''(x)}{2}V[\xi]+o(V[\xi]).\end{aligned} \tag{34}$$

Here, it should be noted that $O(\cdot)$ means the terms of order at most and $o(\cdot)$ means the terms of order smaller than $(\cdot)$. Then, we can derive from combining the Eqs. (30), (31) and (34) that

$$\lambda(x,\xi) = -\frac{1}{2}\frac{u''(x)}{u'(x)}V[\xi]+o(V[\xi]). \tag{35}$$

If $V[\xi]\to 0$, we can get straightforwardly that

$$\lambda(x,\xi) \approx -\frac{1}{2}\frac{u''(x)}{u'(x)}V[\xi], \tag{36}$$

which implies that

$$\lambda(x,\xi) \approx \frac{1}{2}r(x)V[\xi]. \tag{37}$$

For the other cases when the small risk ξ is not actually neutral with $E[\xi]=a$, then based on Eqs. (26), (37), and (22), we obtain that

$$\lambda(x,\xi) = \lambda(x+a,\xi-a) \approx \frac{1}{2}r(x+E[\xi])V[\xi]. \tag{38}$$

Finally, on the basis of the above deduction, the following proposition provides the conclusion of our discussion so far. The proof is omitted as rather straightforward (see also previous discussion).

Proposition 1 Assume that the utility function u is twice differentiable, strictly concave and increasing, and ξ is an LR fuzzy interval. Then the crecibilistic risk premium can be approximately calculated as

$$\lambda(x, \xi) \approx \frac{1}{2} r(x + E[\xi]) V[\xi], \tag{39}$$

where $E[\xi]$ and $V[\xi]$ are the expected value and the small variance of the uncertain risk ξ calculated with credibility measurement, respectively.

3.3. *Comparative risk aversion under credibility measure*

In the field of risk aversion, beside of analysing the local risk aversion with respect to neutral and small risks, Pratt's theorem characterizes the method that compares risk aversion between two agents globally, i.e., for any risks. In order to compare the risk averse attitudes between the agents under the provided credibility measure, we establish a credibilitic version of Pratt's theorem which reveals the relations among local risk aversion functions $r(x)$, utility functions $u(x)$, and credibilistic risk premiums $\lambda(x, \xi)$. The following theorem summarizes the main theoretical findings of our paper.

Theorem 7 Let r_i and $\lambda_i(x, \xi)$ be the local risk aversion function and the credibilistic risk premiums with respect to utility functions u_i, $i = 1, 2$, respectively, where u_i, $i = 1, 2$, are twice differentiable, strictly concave, and strictly increasing. The following propositions are equivalent:
(i) $r_1(x) \geq r_2(x)$ for any x;
(ii) $u_1(u_2^{-1}(x))$ is concave;
(iii) $\lambda_1(x, \xi) \geq \lambda_2(x, \xi)$ for any x and ξ.

Proof. *(i)* $\Rightarrow$ *(ii)*. It can be deduced from Pratt's theorem directly.

(ii) $\Rightarrow$ *(iii)*. According to the definition of credibilistic risk premium (see Definition 9), for any given fuzzy variable ξ, we have

$$\lambda_i(x, \xi) = x + E[\xi] - u_i^{-1}(E[u_i(x + \xi)]),\ i = 1, 2. \tag{40}$$

Then, we take the difference

$$\begin{aligned} \lambda_1(x, \xi) - \lambda_2(x, \xi) &= u_2^{-1}(E[u_2(x + \xi)]) - u_1^{-1}(E[u_1(x + \xi)]) \\ &= u_2^{-1}(E[\eta]) - u_1^{-1}(E[u_1(u_2^{-1}(\eta))]), \end{aligned} \tag{41}$$

where $\eta = u_2(x + \xi)$. In addition, since $u_1(u_2^{-1}(x))$ is concave, based on Corollary 1, we get

$$E[u_1(u_2^{-1}(\eta))] \leq u_1(u_2^{-1}(E[\eta])). \tag{42}$$

Substituting Eq. (42) into Eq. (41), the condition *(iii)* is obtained.

(iii) $\Rightarrow$ *(i)*. Assume that there exists a point x that satisfies $r_1(x) < r_2(x)$, and some small risks with $V[\xi] > 0$ and $E[\xi] = 0$. Then, for these uncertain risks, the local risk premium $\lambda_1(x, \xi) > \lambda_2(x, \xi)$ can be immediately obtained through Eq. (37), which violates the condition *(iii)*. Thus *(iii)* $\Rightarrow$ *(i)* holds. □

Furthermore, letting $u_1(x) = u(x)$ and $u_2(x) = u(x + k)$ for arbitrary x and $k > 0$, the Corollary 2 can be derived through Theorem 7 as follows. The proof follows the previous theorem proof, so it is omitted.

Corollary 2 The following propositions are equivalent.
(i') $r(x)$ is decreasing;
(ii') $u'(u^{-1}(x))$ is convex;
(iii') $\lambda(x,\xi)$ decreases with x for any ξ.

4. Numerical Examples

In order to be able to appreciate more the derived results, and to understand deeper the framework under which the proposed credibilistic risk aversion theory is proposed, and it can be used, the following three numerical examples are presented.

Example 3: Assume that there exist an antique collector with personal assets of 150 (thousands) US$, who wants to sell his *antique vase*. In order to have a better estimation for the potential value of the vase, he is consulting several antique experts. After a careful deliberation, the experts provide to him the possible selection of prices: 10, 15, 20, 25, 35 in (thousands)US$, together with their corresponding possibilities based on their experiences. The respective possibilities are given based on the principle that assigns 1 to the most possible price, and assigns others accordingly (see Fig. 1). In addition, assume that the utility function of the collector is given by $u(x) = 1 - e^{-\frac{x}{20}}, x \in \Re$, where the unit of measure for the assets x is 1 (thousand) US$. The question is proposed to be: *"what is the smallest amount of money for the antique vase that the collector is willing to sell it?"*

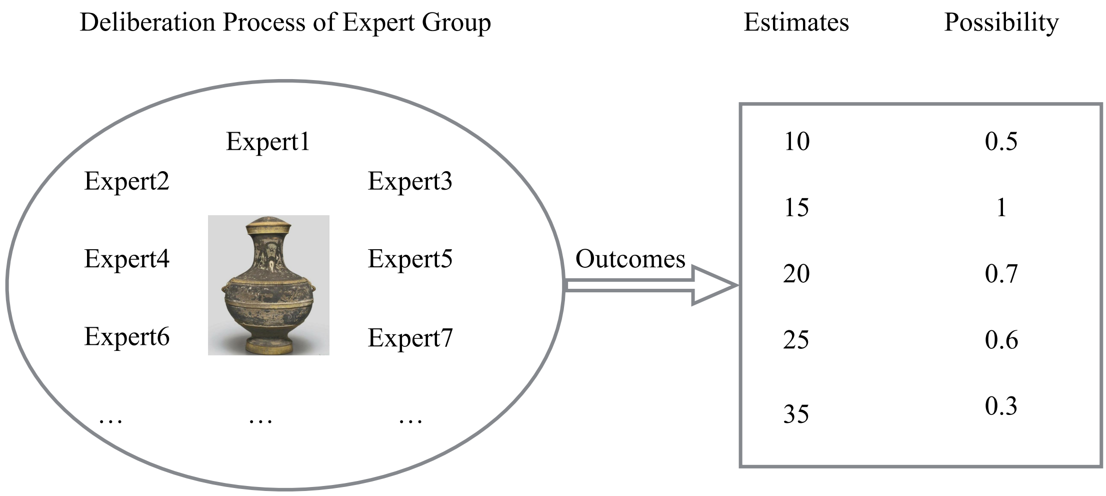

Figure 1. Price estimates of the vase from the expert group

Now, let us denote the possible value of the antique vase as a set of fuzzy variable, which is a discrete one, and it is given by $\xi = \{(10, 0.5), (15, 1), (20, 0.7), (25, 0.3), (35, 0.4)\}$. The value of $\lambda_a(x,\xi)$ in Eq. (28) is exactly what is needed to be calculated. Since the uncertain risk is a discrete fuzzy variable, and its variance is not small, the calculation for local risk premium is not available. Thus, we compute the value of $\lambda_a(x,\xi)$ through a general inverse function operation on Eq. (28), that is, $\lambda_a(x,\xi) = u^{-1}\big(E[u(x+\xi)]\big) - x$, which is similar with the computation of $\lambda(x,\xi)$ in Eq. (27).

(1) First, denote $\eta = u(x+\xi)$. Then η is also a discrete fuzzy variable. Based on the definition of the credibility distribution of a fuzzy variable, the credibility distribution of η can be

deduced as

$$\Psi_\eta(x) = \begin{cases} 0, & \text{if } x < 1 - e^{-\frac{150+10}{20}} \\ 0.25, & \text{if } 1 - e^{-\frac{150+10}{20}} \leq x < 1 - e^{-\frac{150+15}{20}} \\ 0.65, & \text{if } 1 - e^{-\frac{150+15}{20}} \leq x < 1 - e^{-\frac{150+20}{20}} \\ 0.7, & \text{if } 1 - e^{-\frac{150+20}{20}} \leq x < 1 - e^{-\frac{150+25}{20}} \\ 0.85, & \text{if } 1 - e^{-\frac{150+25}{20}} \leq x < 1 - e^{-\frac{150+35}{20}} \\ 1, & \text{otherwise.} \end{cases}$$

After that, according to Eq. (17), the expected value of η can be obtained as 0.99976.

(2) Secondly, since $u(x) = 1 - e^{-\frac{x}{20}}$, we get its inverse function $u^{-1}(x) = -20\ln(1-x),\ x < 1$.

(3) Finally, $\lambda_a(x,\xi) = u^{-1}\big(E[u(x+\xi)]\big) - x = -20\ln(1-0.99976) - 150 = 166.9719 - 150 = 16.9719$ (thousands) US\$. $\lambda_a(x,\xi)$ is exactly the cash equivalent for this antique vase, which means the lowest price that the antique collector is willing to sell the vase. Moreover, it can be easily deduced that the credibilistic risk premium $\lambda(x,\xi) = E[\xi] - \lambda_a(x,\xi) = 18.5 - 16.9719 = 1.5281$.

Example 4: Assume an agent has 10 (thousands) US$ of idle funds and wants to invest them into the stock market. Assume that a stock has been selected. Besides, it is known that the present price of the selected stock is 4 US$ per share, and the price after a month is estimated to be a symmetrical triangular fuzzy variable ξ. Its membership function μ_ξ is depicted in Fig. 2.

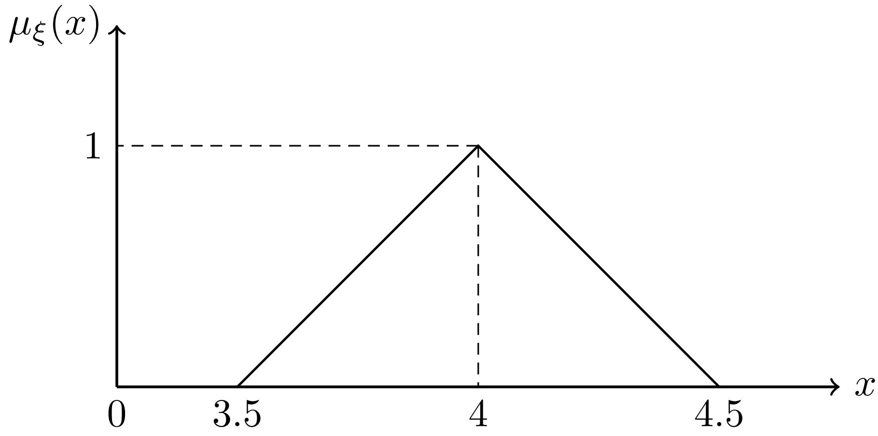

Figure 2. The membership function of ξ in Example 2

Furthermore, except the idle funds, we assume that this investor owns 100 (thousands) US$ assets, and his utility function is also given by $u(x) = 1 - e^{-\frac{x}{20}}, x \in \Re$. Now, the question is: "*Does he prefer to invest the idle funds on this stock or seek for other investment?*"

Denote the uncertain payment of the stock as $\eta = \dfrac{10}{4}\xi$. Then, we need to calculate the cash equivalent of the stock, and make a comparison with the initial fund, i.e. the 10 (thousands) US$. Noticing that ξ is an LR fuzzy interval, and the range of the fluctuate of ξ is relative small, we try to use the method we introduce in Section 3.2 to compute the approximate local risk premium for the stock, and then derive the corresponding cash equivalent. From Eq. (39), we need to obtain the expected value and the variance of η, which is directly derived through the expected value, and the variance of ξ following Eqs. (19) and (22).

First, based on the membership function of ξ described in Fig. 2, and Eq. (11), its credibility distribution can derive directly as

$$\Psi_\xi(x) = \begin{cases} 0, & \text{if } x < 3.5 \\ x - 3.5, & \text{if } 3.5 \le x < 4.5 \\ 1, & \text{otherwise,} \end{cases}$$

which is depicted in Fig. 3.

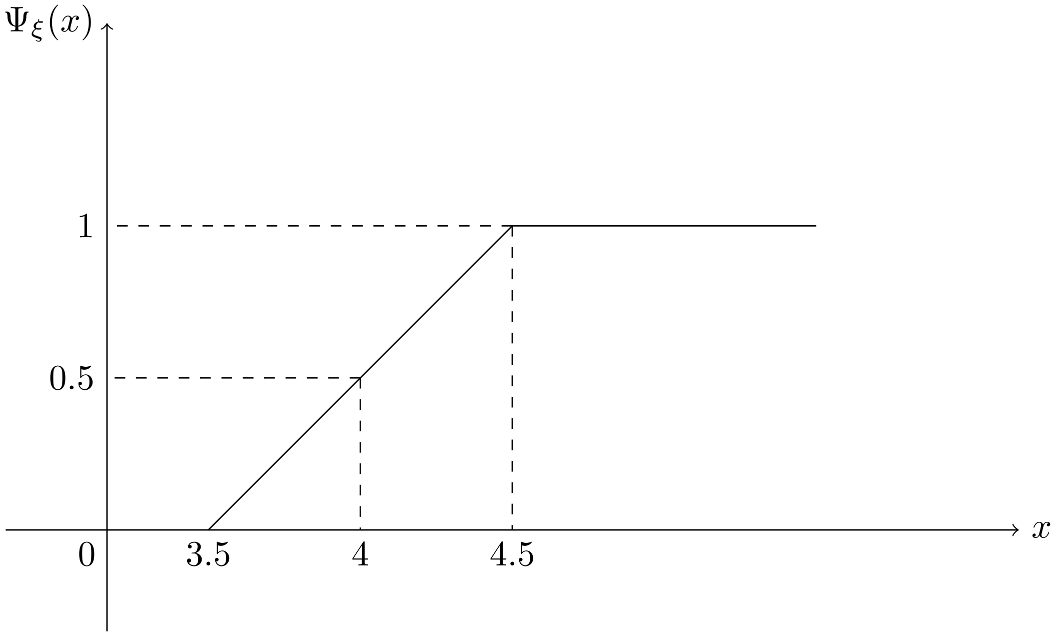

Figure 3. The credibility distribution of ξ

Secondly, through Eq. (17), the expected value of the stock price ξ can be calculated to be equal to 4. Next, we can derive the inverse credibility distribution of ξ as follows

$$\Psi_\xi^{-1}(x) = 3.5 + \alpha, \quad 0 \le \alpha \le 1. \tag{43}$$

Then, the variance of ξ is obtained by Eqs. (21) and (43)

$$\begin{aligned} V[\xi] &= \int_0^1 (\Psi_\xi^{-1}(\alpha) - E[\xi])^2 \mathrm{d}\alpha \\ &= \int_0^1 (3.5 + \alpha - 4)^2 \mathrm{d}\alpha \\ &= \frac{1}{12}. \end{aligned}$$

Based on Eqs. (19) and (22), we know that $E[\eta] = \dfrac{10}{4}E[\xi] = 10$ and $V[\eta] = \left(\dfrac{10}{4}\right)^2 V[\xi] = \dfrac{25}{48}$. Besides, it is known that the local risk aversion function $r(x) = \dfrac{1}{20}$. Thus, on the basis of Eq.

(39), we have $\lambda(x,\eta) = \dfrac{1}{2}r(x+E[\eta])V[\eta] = \dfrac{1}{40} \times \dfrac{25}{48} \approx 0.0130$, and the cash equivalent $\lambda_a(x,\eta) = E[\eta] - \lambda(x,\eta) \approx 10 - 0.0130 = 9.9870$ (thousands) US$.

From the outcomes, we can see that although the largest payment of investing on the stock is $\dfrac{10}{4} \times 4.5 = 11.25$ (thousands) US$, which is much larger than the initial fund of 10 (thousands) US$, the expected value of the uncertain payment with respect to the stock is only 10 (thousands) US$, which is just equal to the initial investment. There is little chance that the agent can make profits from this investment. Due to the attitude of risk aversion of the agent, we can infer the agent will not invest the money on the stock without computing the value of the local risk premium.

Example 5: Assume that the utility function of an agent with initial wealth 20 (thousands) US$ is $u(x) = 1 - \dfrac{100}{(x+10)^2}$, which is a strictly concave function and the unit of measure for the assets x is 1 (thousand) US$. He plans to invest the extra 10 (thousands) US$ (not included in the initial 20 (thousands) US$) into the stock or the bond market. Assume the present price of the stock he selects is 4 US$ per share, and the estimated price after one month is a trapezoidal fuzzy variable ξ with the membership function depicted in Fig. 4, while the bond would return back to him with a certain profit 500 US$. The questions are: (1) *"Will he prefer to invest the money on the stock or the bond?"* (2) *"Will he change his decision when his wealth increases?"*

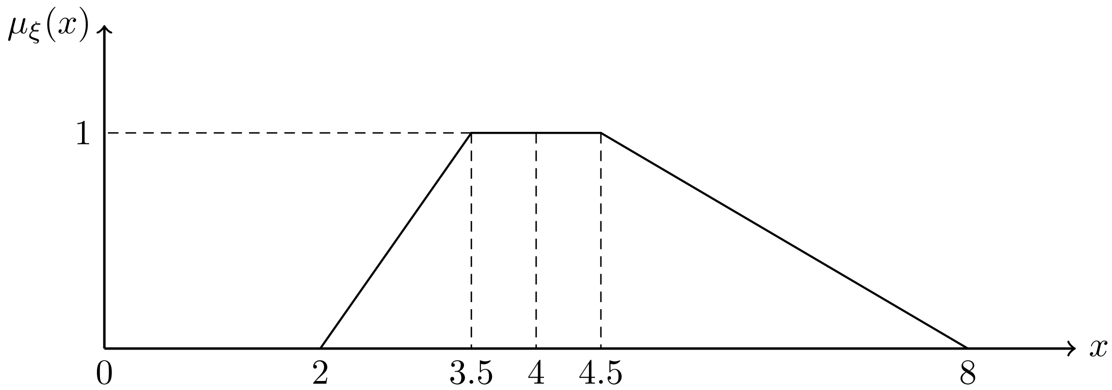

Figure 4. The membership function of ξ

According to his estimation, he believes that the prospective price will fall into the interval $[2, 8]$. And it is the most possible that the price fluctuates between 3.5 and 4.5. Furthermore, comparing with the range of fall, he deems that there is still a significant scope for the selected stock in the next month.

Similar with the problem in Example 4, we denote the uncertain payment in the case of investing the money on the stock as $\eta = \dfrac{10}{4}\xi$. But the fluctuated rang of the stock price is relative big, which will result in a big variance of η too. Thus, we need to utilize the inverse function operation on the utility function in Eq. (28) to calculate the cash equivalent of the payment of investing the stock directly rather than employing the simple computation of the local risk premium.

In order to calculate $\lambda_a(x,\eta)$, the expected value of η and $u(x+\eta)$, i.e., $E[\eta]$ and $E[u(x+\eta)]$ and the inverse function of utility function, i.e., $u^{-1}(x)$, need to be derived.

First, based on the membership function of ξ described in Fig. 5, its credibility distribution can be derived directly as

$$
\Psi_\xi(x) = \begin{cases} 0, & \text{if } \ x < 2 \\ \dfrac{x-2}{3}, & \text{if } \ 2 \leq x < 3.5 \\ 0.5, & \text{if } \ 3.5 \leq x < 4.5 \\ \dfrac{x-1}{7}, & \text{if } \ 4.5 \leq x < 8 \\ 1, & \text{otherwise,} \end{cases}
$$

which is depicted in Fig. 5. Next, the expected value of the stock price ξ after one month is

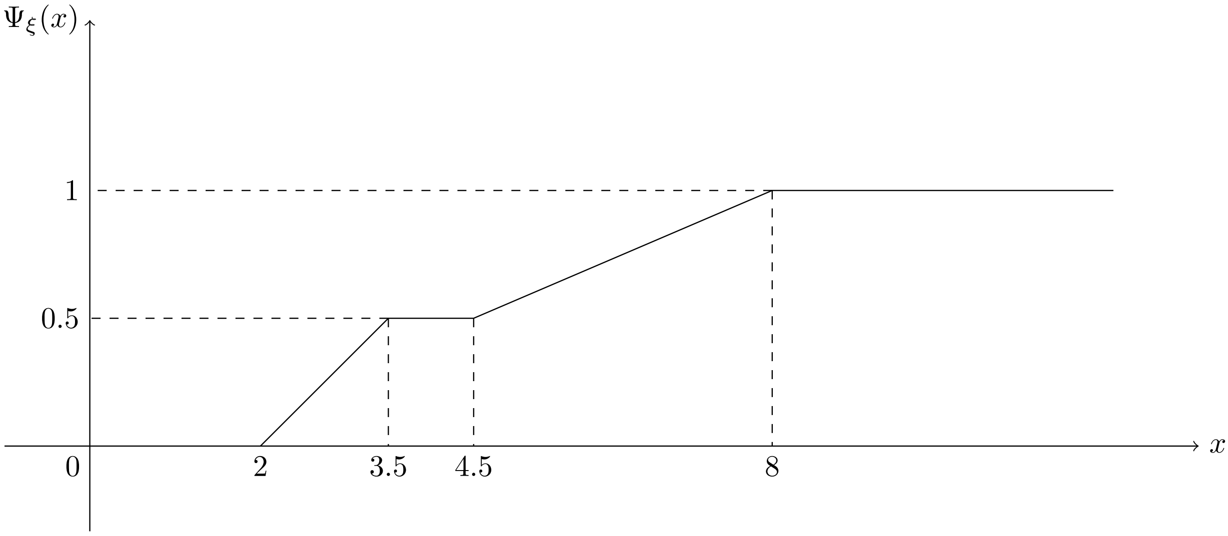

Figure 5. The credibility distribution of ξ

calculated as $\dfrac{9}{2}$, and $E[\eta] = E\left[\dfrac{10}{4}\xi\right] = \dfrac{10}{4} \times \dfrac{9}{2} = \dfrac{45}{4} = 11.25$.

Secondly, according to the definition of the inverse function for LR fuzzy intervals, the inverse credibility distribution of ξ can be deduced as

$$
\Psi_\xi^{-1}(x) = \begin{cases} 3\alpha + 2, & \text{if } \ 0 \leq \alpha < 0.5 \\ a, & \text{if } \ \alpha = 0.5 \\ 7\alpha + 1, & \text{if } \ 0.5 < x < 1, \end{cases} \tag{44}
$$

where a is any value in the interval $[3.5, 4.5]$.

Finally, since the utility function is a strictly increasing function of $x + \eta$, the expected value

$E[u(x+\eta)]$ can be calculated based on Eqs. (16) and (18) as

$$
\begin{aligned}
E[u(x+\eta)] &= E[u(x+\frac{10}{4}\xi)] = \int_0^1 u(x+\frac{10}{4}\Psi^{-1}(\alpha))\mathrm{d}\alpha \\
&= \int_0^{0.5} u(x+\frac{10}{4}\Psi_\xi^{-1}(\alpha))\mathrm{d}\alpha + \int_{0.5}^1 u(x+\frac{10}{4}\Psi_\xi^{-1}(\alpha))\mathrm{d}\alpha \\
&= \int_0^{0.5} 1 - \frac{100}{(x+\frac{10}{4}(3\alpha+2)+10)^2}\mathrm{d}\alpha + \int_{0.5}^1 1 - \frac{100}{(x+\frac{10}{4}(7\alpha+1)+10)^2}\mathrm{d}\alpha \\
&= 1 - \frac{200}{(4x+75)(x+15)} - \frac{200}{(4x+85)(x+30)}.
\end{aligned}
$$

Next, we obtain the inverse of the utility function u, i.e. $u^{-1}(x) = \sqrt{\frac{100}{1-x}} - 10$ for $x < 1$.

Finally, the cash equivalent of the payment of investing the money on the stock is a function of his initial wealth as

$$
\begin{aligned}
\lambda_a(x,\eta) &= u^{-1}(E[u(x+\eta)]) - x \\
&= \sqrt{\frac{100}{1-E[u(x+\eta)]}} - 10 - x \\
&= \sqrt{\frac{(4x+75)(x+15)(4x+85)(x+30)}{16x^2+680x+7350}} - x - 10.
\end{aligned} \tag{45}
$$

For the agent, in our example, whose initial wealth is 20 (thousands) US\$, we can get the cash equivalent $\lambda_a(x,\eta) = 10.4528$ (thousands) US\$, which is lower than the certain payment 10.5 (thousands) US\$ from investing the money on bonds. Since the utility function of the agent is a strictly increasing function, that is the utility taking from the certain 10.5 (thousands) US\$ is larger than a cash equivalent of 10.4528 (thousands) US\$. Therefore, answering the first question, we can infer that he prefers to invest his money on the bond instead of the stock market.

Moreover, regarding the second question, a further analysis on the function of the cash equivalent of η in Eq. (45) is needed. Recall that the utility function is $u(x) = 1 - \frac{100}{(x+10)^2}$, and the local risk aversion function is $r(x) = \frac{3}{x+10}$, which is a decreasing function of x. So, according to Corollary 2, the risk premium $\lambda(x,\eta)$ is also a decreasing function of x for all η including the uncertain payment η in our problem, which means with the increasing of the wealth x of the agent, the risk premium with respect to η decreases, and the cash equivalent of η increases that can also derive from the direct analysis of Eq. (45). When the cash equivalent λ_a reaches up to the certain payment from the bond, the agent will change his decision. Through solving Eq. (45) with $\lambda_a(x,\eta) = 10.5$, we can get that the value of x is equal to 22.7087 (thousands) US\$. When the wealth of the agent is more than 22.7087 (thousands) US\$, he is more willing to take the risk to invest his money on the stock rather than getting a certain earning from the bond.

Subsequently, we depict the function of $\lambda_a(x,\eta)$ in Eq. (45) and the utility function of x in Fig. 6, which could give an intuitive understanding for the case in our problem.

The solid line represents the utility function of x, and the line marked with "+" represents the changes of the cash equivalent of η with the changes of x. It can be observed that both functions of x are strictly increasing. At the critical point, the cash equivalent of η is equal to the certain payment of the bond. The point labelled with "$\star$" indicates the present cash equivalent 10.4258 (thousands) US\$ with the wealth of 20 (thousands) US\$. If and only if the wealth is bigger than the value corresponding to the critical point, i.e., 22.7087 (thousands) US\$, the cash

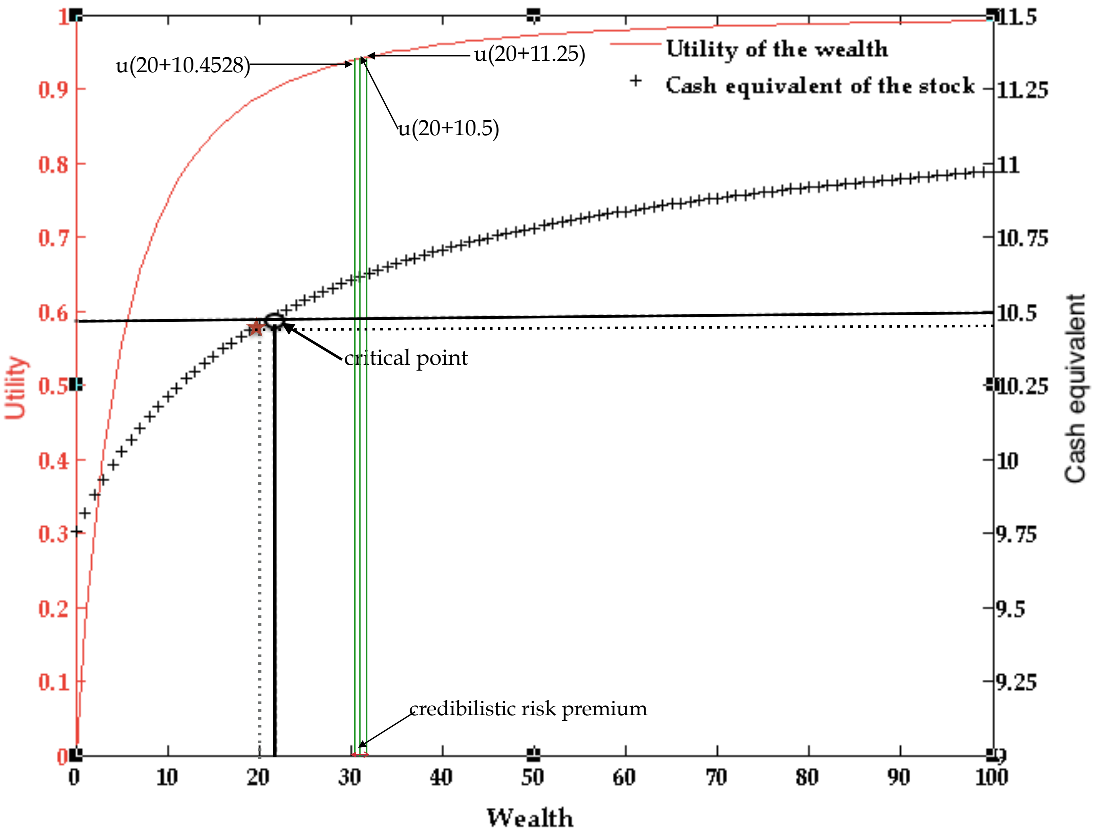

Figure 6. The utility function and the cash equivalent of the wealth x

equivalent of the stock is bigger than the certain payment from the bond. Besides, the utilities of $20 + 10.4528 = 30.4528$ (thousands) US$ (initial wealth with the present cash equivalent of η), 20+10.5 (initial wealth with the certain earning from the bond), and $20+11.25 = 31.25$ (thousands) US$ (initial wealth with the expected value of η) are labelled, respectively. And the credibilistic risk premium of η can be obtained as the difference of the cash equivalent and the expected value of η, i.e., $11.25 - 10.4528 = 0.7972$ (thousands) US$.

Remark 1 It should be noted here that the above numerical examples are simple illustrations for the decision makers in order to appreciate the theoretical findings, to understand the cases that our proposed crecibilistic risk aversion approach can be applied for, and to solve and analyse the economic decision-makings. Both the utility function and the fuzzy variables representing the risks in our examples can be generalized according to the demand and the real world application.

5. Conclusion

The traditional probabilistic risk aversion approach can not deal with all the cases, for instance, when the distribution of the facing risk is tough to be measured or it varies with the decision makers' own deliberation, an alternative credibilistic approach has been suggested instead in order to analyse the risk aversion problems. Thus, in this paper, we put forward the new concept of the credibilistic risk premium. Its relationship with the variance of the uncertain risk represented with a fuzzy variable, and the local risk aversion function $r(x)$ were provided. Subsequently, a comparative risk aversion was studied which can be used to interpret the different economic behaviours. We also enumerated three practical examples in which the probabilistic risk aversion is ineffective, and we show how to utilize the proposed credibilistic risk aversion in order to calculate the credibilistic risk premium, and assist the agent to make the optimal strategy.

Finally, the proposed results can be further extended. As an ongoing project, by using different types of fuzzy variables to express the risk, it can be investigated how to make a decision when the agent confronts several uncertain risks or how to distinguish the utility function based on the economic behaviour of the agent or considering the risk is an multi-period one. For the new direction, research work is in progress.

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